## WREATH PRODUCT DECOMPOSITIONS FOR TRIANGULAR MATRIX SEMIGROUPS

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ABSTRACT. We consider wreath product decompositions for semigroups of triangular matrices. We exhibit an explicit wreath product decomposition for the semigroup of all  $n \times n$  upper triangular matrices over a given field k, in terms of aperiodic semigroups and affine groups over k. In the case that k is finite this decomposition is optimal, in the sense that the number of group terms is equal to the group complexity of the semigroup. We also obtain some decompositions for semigroups of triangular matrices over more general rings and semirings.

#### 1. Introduction

Some of the most natural and frequently occurring semigroups are those of upper triangular matrices over a given ring or field. For example, such semigroups arise in the study of algebraic semigroups, where Putcha [17] has proven that a connected algebraic monoid with zero over a field has a faithful rational triangular representation if and only if its group of units is solvable [17]. It follows that triangularizable monoids can be thought of as a natural generalisation of solvable groups. More recently, Almeida, Margolis and Volkov [1] have shown that semigroups of triangular matrices over finite fields generate natural pseudovarieties. Almeida, Margolis, Steinberg and Volkov [2, 3] have since considered arbitrary fields and have obtained language-theoretic consequences. Further properties of these semigroups have been described by Okninski [16].

Perhaps the most productive approach to the study of finite semigroups is through coverings by wreath products. In the 1960s, Krohn and Rhodes [12, 13, 14] showed that every finite semigroup can be expressed as a *divisor* (a homomorphic image of a subsemigroup) of a wreath product of finite groups and finite aperiodic monoids. The *group complexity* of a finite semigroup is the smallest number of group terms in such a decomposition, and is a key concept in finite semigroup theory.

In a previous article [11], the first author computed the group complexity of the semigroup  $T_n(k)$  of all  $n \times n$  upper triangular matrices over a given finite field k, and of certain related semigroups. However, the methods used did not result in explicit wreath product decompositions. The main objective of this article is to establish an explicit wreath product decomposition for each semigroup of the form  $T_n(k)$ , and hence for every semigroup of triangular matrices over a finite field. This decomposition is optimal, in the sense that the number of group terms in the decomposition is equal to the group complexity of  $T_n(k)$ . Moreover every group appearing is a product of subgroups of  $T_n(k)$ .

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In the process, we obtain some results applicable in a more general context. While Krohn-Rhodes theory is traditionally concerned with finite semi-groups, there have been numerous attempts to extend it to well-behaved classes of infinite semigroups [4, 6, 8]. Our method for decomposing  $T_n(k)$  is fully applicable in the case that the field k, and hence also the semigroup  $T_n(k)$ , is infinite. We also obtain some wreath product decompositions, although not in terms of groups and aperiodic semigroups, for triangular matrix semigroups over more general rings and semirings with identity.

In addition to this introduction, this paper comprises four sections. In Section 2, we briefly recall the key definitions and results of Krohn-Rhodes theory as applied to abstract monoids, including division, wreath products, the Prime Decomposition Theorem and group complexity. Section 3 introduces triangular matrix semigroups, and briefly describes their structure, before reviewing the results of the first author [11] characterising their group complexity.

Section 4 contains the main original results of the paper; we obtain an explicit decomposition for each semigroup  $T_n(k)$ , and hence for every semigroup of triangular matrices over a field, as a wreath product of aperiodic monoids and affine groups over k. We also obtain some related decompositions for triangular matrix semigroups over rings and semirings with identity. Finally, in Section 5, we compare our results with those which can be obtained using a standard decomposition method of Eilenberg and Tilson [20]; the latter produces a suboptimal decomposition for  $T_n(k)$ , but alternative optimal decompositions for certain important divisors.

Throughout this paper, all functions are applied on the right of their arguments. If S and T are sets then we denote by  $S^T$  the set of all functions from T to S. We assume familiarity with the standard terminology, notation and foundational results of structural semigroup theory; a detailed introduction to these is given by Howie [10]. By contrast, we assume no prior knowledge which is particular to the study of finite semigroups; we intend that this article should be fully accessible to the reader with experience only of infinite semigroups.

# 2. Wreath Products, Division and Complexity

In this section, we briefly introduce the basic concepts of wreath products, division and complexity. We restrict ourselves to the special case of abstract monoids (as opposed to transformation semigroups), since this suffices for our purpose. A detailed and more general introduction is given by Eilenberg [5].

Let S and T be semigroups. We say that S divides T, and write  $S \prec T$ , if S is a homomorphic image of some subsemigroup of T. The relation of division is easily verified to be reflexive and transitive.

Let S and T be monoids. Then  $S^T$  is a monoid with pointwise product: if  $f, g \in S^T$  and  $t \in T$ , then by definition t(fg) = (tf)(tg). There is also a natural left action of T on  $S^T$  defined as follows: if  $f \in S^T$ ,  $t_1, t_2 \in T$ , then  $t_1 \in T \to S$  is given by

$$t_2^{t_1}f = (t_2t_1)f.$$

Then the wreath product of S and T, denoted  $S \wr T$ , is the monoid with underlying set  $S^T \times T$ , and multiplication given by

$$(f,a)(g,b) = (f^a g, ab).$$

The wreath product of monoids is not associative; however,  $(S_3 \wr S_2) \wr S_1$  is isomorphic to a submonoid of  $S_3 \wr (S_2 \wr S_1)$ . For this reason, we define the *iterated wreath product* of a sequence of three or more monoids inductively by

$$S_n \wr S_{n-1} \wr \cdots \wr S_1 = S_n \wr (S_{n-1} \wr \cdots \wr S_1)$$

so as to obtain the largest monoid possible.

Recall that a semigroup is called *aperiodic* if it has no non-trivial subgroups. In the following proposition we state without proof a few well-known properties of the wreath product which we shall need.

**Proposition 2.1.** Let A, B, C and D be finite monoids.

- (i) If  $A \prec B$  then  $A \wr C \prec B \wr C$  and  $C \wr A \prec C \wr B$ .
- (ii)  $A \times B \prec A \wr B$ .
- (iii)  $(A \wr B) \times (C \wr D) \prec (A \times C) \wr (B \times D)$ .
- (iv) If A and B are groups then  $A \wr B$  is a group.
- (v) If A and B are aperiodic then  $A \wr B$  is aperiodic.

We shall also need an elementary decomposition that is perhaps not so well known:

**Proposition 2.2.** Let A, B, C be monoids. Then  $(A \wr B) \times C$  embeds in  $A \wr (B \times C)$ .

*Proof.* First we define a homomorphism  $\alpha:A^B\to A^{B\times C}$  by  $(b,c)f\alpha=bf$ . Next we define  $\psi:(A\wr B)\times C\to A\wr (B\times C)$  by

$$((f,b),c)\psi = (f\alpha,(b,c)).$$

Let us verify that  $\psi$  is a homomorphism.

$$((f,b),c)\psi ((g,b'),c')\psi = (f\alpha,(b,c)) (g\alpha,(b',c'))$$
  
=  $(f\alpha^{(b,c)}g\alpha,(bb',cc')).$ 

But for  $(b_0, c_0) \in B \times C$ ,

$$(b_0, c_0)^{(b,c)}g\alpha = (b_0b, c_0c)g\alpha = b_0bg = b_0{}^bg.$$

Thus  $(b,c)(g\alpha) = (bg)\alpha$ . Hence we may conclude

$$((f,b),c)\psi ((g,b'),c')\psi = (f\alpha({}^{b}g)\alpha,(bb',cc'))$$
$$= ((f^{b}g)\alpha,(bb',cc'))$$
$$= [((f,b),c)((g,b'),c')]\psi$$

and so  $\psi$  is a homomorphism. It is clear that  $\psi$  is injective.

Let X be a finite set. We denote by  $\widetilde{X}$  the monoid consisting of the identity map and all constant maps on X; clearly,  $\widetilde{X}$  is an aperiodic monoid. Now if A is a monoid of transformations of X, then the *augmented monoid*  $\overline{A}$  of A with respect to its action on X is the monoid generated by transformations in A and those in  $\widetilde{X}$ . The following proposition, a proof of which

can be found in Eilenberg [5], provides a decomposition of an augmented monoid in terms an aperiodic monoid and the underlying monoid.

**Proposition 2.3.** Let A be a finite monoid of transformations of a set X. Then  $\overline{A} \prec \widetilde{X} \wr A$ .

The importance of wreath products for the study of finite semigroups stems from the following structure theorem of Krohn and Rhodes [12, 14].

**Theorem 2.4.** (The Prime Decomposition Theorem, Krohn-Rhodes 1968) Let S be a finite semigroup. Then S divides some iterated wreath product each of whose terms is either (i) a finite simple group which divides S or (ii) a finite aperiodic monoid.

A Krohn-Rhodes decomposition for a semigroup S is an expression of S as a divisor of an iterated wreath product of groups and aperiodic monoids. Given such a decomposition for S, Proposition 2.1 tells us that we can combine adjacent groups terms and adjacent aperiodic terms to obtain an alternating decomposition of the form:

$$A_n \wr G_n \wr A_{n-1} \wr \cdots \wr A_1 \wr G_n \wr A_0$$

where each  $A_i$  is aperiodic, each  $G_i$  is a group, and all terms except possibly  $A_0$  and  $A_n$  are non-trivial. (Note, though, that in doing so we may lose the property that the group terms are divisors of S.) The number n, that is, the number of group terms, is called the *group length* of the decomposition. A natural structural constant which can be associated with a finite semigroup S is the minimal group length of a Krohn-Rhodes decomposition for S; this number is called the *group complexity* of S. A decomposition for S is said to be *optimal* if its group length equals the group complexity of S.

Much effort has been put into the study of Krohn-Rhodes decompositions, and in particular of certain algorithmic problems. Various algorithms have been developed for finding wreath product decompositions for semigroups; some of these will be discussed in Section 5 below. A major open question is that of whether group complexity is decidable, that is, whether there is an algorithm which, given the multiplication table for a finite semigroup S, determines the group complexity of S.

We remark briefly upon the relationship between these two problems, and in particular on the implications of the latter for the former. In theory, knowing the group complexity of a finite semigroup allows one to compute an optimal decomposition. Indeed, if ones knows that a semigroup S admits a decomposition of group length n, then one can in principle enumerate multiplication tables of divisors of alternating wreath products with n group terms, and test them for isomorphism with S. In practice, of course, this algorithm is completely infeasible – the cardinality of an iterated wreath product grows extremely fast as function of the cardinalities of the terms, and no sensible upper bounds are known even on the latter. Hence, situations can arise in which the complexity of a semigroup is known, but an explicit optimal decomposition is not. Indeed, the following key result of Rhodes [19] often gives rise to such situations.

**Theorem 2.5.** (The Fundamental Lemma of Complexity, Rhodes 1974) Let S and T be finite semigroups, and suppose there exists a surjective morphism

 $S \to T$  which is injective when restricted to each subgroup of S. Then S and T have the same group complexity.

The Fundamental Lemma is an extremely powerful tool for computing the group complexity of a semigroup. The proof of the Lemma given by Tilson [21] is constructive in the sense that, given an optimal wreath product decomposition for a semigroup T and a surjective morphism  $S \to T$  which is injective on subgroups, it does provide an optimal decomposition for S. However the construction is quite involved and in practice it is hard to see what groups and aperiodic monoids appear.

## 3. Triangular Matrix Semigroups

Let R be a semiring with identity 1 and zero 0. If x is an  $n \times n$  matrix then for  $1 \le i, j \le n$  we denote by  $x_{ij}$  the entry of x in position (i, j), that is, in the ith row and jth column, of x. Recall that the matrix x is (upper) triangular if  $x_{ij} = 0$  whenever  $1 \le j < i \le n$ . We call an upper triangular matrix (upper) unitriangular if, in addition,  $x_{ii} = 0$  or  $x_{ii} = 1$  for  $1 \le i \le n$ . We call x a subidentity if it is unitriangular and  $x_{ij} = 0$  whenever  $i \ne j$ . We denote by  $T_n(R)$  and  $UT_n(R)$  the semigroups of all  $n \times n$  upper triangular matrices and of all  $n \times n$  unitriangular matrices respectively, with entries drawn from R, the operation in both cases being usual matrix multiplication. Note that  $T_1(R)$  is just the multiplicative semigroup of R.

We shall be especially interested in the case that the semiring R is a field k. In this case, we define a relation  $\sigma$  on each semigroup  $T_n(k)$  by  $x \sigma y$  if and only  $x = \lambda y$  for some non-zero scalar  $\lambda$ . This relation is easily verified to be a congruence on  $T_n(k)$ . The projective triangular semigroup  $PT_n(k)$  is the quotient semigroup  $T_n(k)/\sigma$ ; we denote by  $\overline{x}$  the element of  $PT_n(k)$  which is the  $\sigma$ -equivalence class of a matrix  $x \in T_n(k)$ .

The group of units of  $T_n(k)$  [respectively,  $UT_n(k)$ ,  $PT_n(k)$ ] is denoted  $T_n^*(k)$  [ $UT_n^*(k)$ ,  $PT_n^*(k)$ ]. It consists of those triangular matrices whose diagonal entries are non-zero [respectively, triangular matrices whose diagonal entries are 1, equivalence classes of triangular matrices whose diagonal entries are non-zero]. Note that  $T_1^*(k)$  is the multiplicative group of the field k.

We introduce a notion of upper triangular row and column operations on  $T_n(R)$  and hence on  $UT_n(R)$ . By a row operation on an upper triangular matrix we shall mean either (i) adding a multiple of one row to a row above or (ii) scaling a row by an element of R. There is an obvious analogous definition of column operations of different types, a type (i) operation being adding a multiple of one column to a column to the right. The following easy proposition characterises Green's relations in  $T_n(R)$  and  $UT_n(R)$  in terms of these operations.

**Proposition 3.1.** Let n be a positive integer and R a semiring with identity. Two matrices in  $T_n(R)$  [respectively,  $UT_n(R)$ ] are:

- (i) L-related exactly if each can be obtained from the other by [unitriangular] row operations;
- (ii) R-related exactly if each can be obtained from the other by [unitri-angular] column operations;

(iii) *J*-related exactly if each can be obtained from the other by [unitriangular] row and column operations.

We now turn our attention to the case of a finite field k. The following proposition, parts of which go back at least as far as Putcha [17], characterises the regular elements in  $T_n(k)$ . A proof can be found in a previous article of the first author [11].

**Proposition 3.2.** Let n be a positive integer and k a finite field. Let  $x \in T_n(k)$  or  $x \in UT_n(k)$ . Then the following are equivalent:

- (i) x is regular;
- (ii) every row in x is a linear combination of rows in x with non-zero diagonal entries;
- (iii) every column in x is a linear combination of columns in x with non-zero diagonal entries;
- (iv) x is  $\mathcal{J}$ -related to a subidentity.

Factoring out a monoid by a subgroup of the group of units that is central in the monoid gives rise to a congruence contained in  $\mathcal{H}$ . The following simple observation is a special case of well-known and elementary facts about congruences contained in  $\mathcal{H}$ .

**Proposition 3.3.** Let n be a positive integer, k a field and  $x, y \in T_n(k)$ . Then

- (i) x is regular in  $T_n(k)$  if and only if  $\overline{x}$  is regular in  $PT_n(k)$ ;
- (ii)  $x \mathcal{L} y$  in  $T_n(k)$  if and only if  $\overline{x} \mathcal{L} \overline{y}$  in  $PT_n(k)$ ;
- (iii)  $x \mathcal{R} y$  in  $T_n(k)$  if and only if  $\overline{x} \mathcal{R} \overline{y}$  in  $PT_n(k)$ ;
- (iv)  $x \mathcal{J} y$  in  $T_n(k)$  if and only if  $\overline{x} \mathcal{J} \overline{y}$  in  $PT_n(k)$ .

We recall the following theorem of the first author [11].

**Theorem 3.4.** (Kambites 2004) Let n be a positive integer, and k a finite field. If n > 1 or  $k = \mathbb{Z}_2$  then  $T_n(k)$ ,  $UT_n(k)$  and  $PT_n(k)$  have complexity n-1. If n = 1 and  $k \neq \mathbb{Z}_2$  then  $UT_n(k)$  and  $PT_n(k)$  have complexity 0, while  $T_n(k)$  has complexity 1.

We remark that the scope of this result has since been extended by Mintz [15]; he observes that triangular matrix semigroups form a special class of quiver algebra and that the result extends naturally to cover a somewhat larger class of quiver algebras.

The proof of Theorem 3.4 is somewhat technical, and makes extensive use of the Fundamental Lemma of Complexity, both directly and through the application of a result of Rhodes and Tilson [18]. Consequently, it does not give rise to explicit Krohn-Rhodes decompositions for the semigroups in question. In the next section, we shall show how to obtain such decompositions for semigroups of the form  $T_n(k)$ , and hence for every triangular matrix semigroup over a field.

## 4. Decompositions for Triangular Matrix Semigroups

Our main objective in this section is to compute an explicit decomposition for each semigroup of the form  $T_n(k)$  with k a field, as a divisor of an alternating wreath product of groups and aperiodic monoids. In the case

that k is finite, this decomposition will be optimal, in the sense that its group length equals the group complexity of the semigroup as described by Theorem 3.4. In the process, we also obtain some decompositions for triangular matrix semigroups over more general rings and semirings.

Let R be a semiring and n a positive integer. We consider the R-module  $R^n$  of  $1 \times n$  row vectors over R. Recall that an affine transformation of  $R^n$  is a map of the form  $v \mapsto vX + c$  for some  $n \times n$  matrix X and some vector  $c \in R^n$ . We say that the transformation is affine (upper) triangular if X is upper triangular, and affine scaling if X is of the form  $\lambda I$  where  $\lambda \in R$  and I is the identity matrix.

The affine monoid  $A_n(R)$  of degree n over R is the monoid of all affine transformations of  $R^n$ , with operation composition. It is readily verified that the sets of affine triangular and affine scaling maps form submonoids; these we call the affine triangular monoid  $AT_n(R)$  and the affine scaling monoid  $AS_n(R)$  respectively. The affine group  $A_n^*(R)$ , the affine triangular group  $AT_n^*(R)$  and the affine scaling group  $AS_n^*(R)$  are the groups of units of  $A_n(R)$ ,  $AT_n(R)$  and  $AS_n(R)$  respectively. We remark that the various affine groups are semidirect products of the appropriate matrix groups and with the additive group of translations.

There is a natural embedding of an affine triangular monoid of degree n-1 into an upper triangular monoid of degree n.

**Proposition 4.1.** Let  $n \geq 2$  and let R be a semiring. Then  $AT_{n-1}(R)$  and  $AS_{n-1}(R)$  embed in  $T_n(R)$ .

*Proof.* From the definition,  $AS_{n-1}(R)$  is a subsemigroup of  $AT_{n-1}(R)$ , so it suffices to show that the latter embeds in  $T_n(R)$ . Given an affine triangular map f given by  $v \mapsto vX + c$  we define an  $n \times n$  matrix

$$M_f = \begin{pmatrix} 1 & c \\ 0 & X \end{pmatrix}.$$

That the matrix  $M_f$  is upper triangular follows from the fact that X is upper triangular. If we identify  $v \in R^{n-1}$  with (1,v) then it is routine to verify that  $(1,v)M_f = (a,vf)$  and so  $f \mapsto M_f$  gives an embedding of  $AT_{n-1}(R)$  into  $T_n(R)$ , as required.

The following lemma is the main inductive step in our decompositions. If X is a matrix, we write  $X^T$  for its transpose.

**Lemma 4.2.** Let  $n \geq 2$  and R be a semiring with identity. Then

$$T_n(R) \prec [AS_{n-1}(R) \wr T_{n-1}(R)] \times T_1(R).$$

*Proof.* We view each  $s \in T_n(R)$  as a block matrix

$$s = \begin{pmatrix} M_s & v_s \\ 0 & c_s \end{pmatrix}$$

where  $M_s$  is an  $(n-1) \times (n-1)$  matrix which clearly lies in  $T_{n-1}(R)$ ,  $v_s$  is an  $n \times 1$  column vector and  $c_s$  is a  $1 \times 1$  matrix. Now we define

$$\psi: T_n(R) \to [AS_{n-1}(R) \wr T_{n-1}(R)] \times T_1(R)$$

by

$$s\psi = (f_s, M_s, c_s)$$

where for every  $M \in T_{n-1}(R)$ , the element  $Mf_s \in AS_{n-1}(R)$  is given by

$$w(Xf_s) = (Xv_s + w^T c_s)^T.$$

Clearly,  $\psi$  is well-defined; it is also injective, since for any s, we have  $v_s = [\underline{0}(If_s)]^T$  where  $I \in T_{n-1}(R)$  is the identity matrix and  $\underline{0} \in R^{n-1}$  is the zero vector.

To prove the lemma, it will now suffice to show that  $\psi$  is a homomorphism. Since

$$\begin{pmatrix} M_s & v_s \\ 0 & c_s \end{pmatrix} \begin{pmatrix} M_t & v_t \\ 0 & c_t \end{pmatrix} = \begin{pmatrix} M_s M_t & M_s v_s + v_s c_t \\ 0 & c_s c_t \end{pmatrix},$$

we have:  $M_{st} = M_s M_t$ ,  $c_{st} = c_s c_t$  and  $v_{st} = M_s v_t + v_s c_t$ . So, recalling the definition of the wreath product, it remains to show that  $f_{st} = f_s^{M_s} f_t$ . That is we must show  $w(X f_{st}) = w[(X f_s)(X M_s f_t)]$  for all  $w \in \mathbb{R}^n$  and  $X \in T_{n-1}(\mathbb{R})$ . But

$$w(Xf_{st}) = (Xv_{st} + w^{T}c_{st})^{T}$$

$$= (X(M_{s}v_{t} + v_{s}c_{t}) + w^{T}c_{s}c_{t})^{T}$$

$$= (XM_{s}v_{t} + Xv_{s}c_{t} + w^{T}c_{s}c_{t})^{T}$$

$$= (XM_{s}v_{t} + (Xv_{s} + w^{T}c_{s})c_{t})^{T}$$

$$= (XM_{s}v_{t})^{T} + (Xv_{s} + w^{T}c_{s})^{T}c_{t}$$

$$= (XM_{s}v_{t})^{T} + (wXf_{s})c_{t}$$

$$= (XM_{s}v_{t} + (w(Xf_{s}))^{T}c_{t})^{T}$$

$$= w[(Xf_{s})(XM_{s}f_{t})]$$

as required.

Lemma 4.2 leads easily to the following decomposition for  $T_n(R)$  in terms of affine scaling monoids and the multiplicative semigroup of R.

**Theorem 4.3.** Let  $n \ge 2$  and R be a semiring with identity. Then

$$T_n(R) \prec AS_{n-1}(R) \wr AS_{n-2}(R) \wr \cdots \wr (AS_1(R) \times T_1(R)^n).$$

*Proof.* We use induction on n. When n=2 then using Lemma 4.2 and Proposition 2.2 we have

$$T_2(R) \prec [AS_1(R) \wr T_1(R)] \times T_1(R) \prec AS_1(R) \wr T_1(R)^2$$

as required. Now let  $n \geq 3$  and assume true for smaller n. Then again using Lemma 4.2 and Proposition 2.2 we obtain

$$T_n(R) \prec [AS_{n-1}(R) \wr T_{n-1}(R)] \times T_1(R)$$

$$\prec [AS_{n-1}(R) \wr (AS_{n-2}(R) \wr \cdots \wr (AS_1(R) \times T_1(R)^{n-1}))] \times T_1(R)$$

$$\prec AS_{n-1}(R) \wr AS_{n-2}(R) \wr \cdots \wr (AS_1(R) \times T_1(R)^n)$$

as required.  $\Box$ 

As a consequence of Theorem 4.3, we obtain a group length n-1 decomposition for each semigroup  $T_n(k)$  with k a field.

**Theorem 4.4.** Let  $n \geq 2$  and k be a field. Then  $T_n(k)$  divides

$$\widetilde{k^{n-1}} \wr AS_{n-1}^*(k) \wr \widetilde{k^{n-2}} \wr AS_{n-2}^*(k) \wr \cdots \wr \widetilde{k} \wr [AS_1^*(k) \times T_1^*(k)^n] \wr U_1^n$$

where  $U_1$  is the two-element semilattice.

*Proof.* By Theorem 4.3 we have that

$$T_n(k) \prec AS_{n-1}(k) \wr AS_{n-2}(k) \wr \cdots \wr (AS_1(k) \times T_1(k)^n).$$

For each i, it is easily seen that the affine monoid  $AS_i(k)$  consists precisely of  $AS_i^*(k)$  and constant maps on  $k^i$ ; hence,  $AS_i(k)$  is the augmented monoid of  $AS_i^*(k)$  with respect to its action on  $k^i$ , and so by Proposition 2.3 we have

$$AS_i(k) \prec \widetilde{k}^i \wr AS_i^*(k).$$

Also, it is easy to see that the group with zero  $T_1(k)$  divides  $T_1^*(k) \times U_1$ . It follows that

$$AS_1^*(k) \times T_1(k)^n \prec AS_1^*(k) \times T_1^*(k)^n \times U_1^n \prec (AS_1^*(k) \times T_1^*(k)^n) \wr U_1^n.$$
  
The result is now clear.

Recall that the *pseudovariety* generated by a finite semigroup S is the class of all divisors of finite direct products of S. In general, a finite semigroup S does not necessarily admit an optimal Krohn-Rhodes decomposition whose group terms are divisors of S, or even in the pseudovariety generated S. Here we have succeeded in finding for  $T_n(k)$  an optimal Krohn-Rhodes decomposition in which every group is a subgroup of the group of units  $T_n(k)$  except one, which is a direct product of two subgroups of  $T_n^*(k)$ . Indeed, Proposition 4.1 implies that each  $AS_m^*(k)$  with  $1 \leq m \leq n-1$  embeds in  $T_n^*(k)$ . On the other hand  $T_1^*(k)^n$  is just the diagonal subgroup of  $T_n^*(k)$ .

### 5. Comparison with Depth Decomposition

Considerable thought has been put into algorithmic methods for obtaining explicit Krohn-Rhodes decompositions for finite transformation semigroups. The original proof of Krohn and Rhodes [13] is essentially algorithmic; however, the decompositions it yields are far from optimal. A substantial improvement is the *holonomy method*, which was developed by Eilenberg [5], in conjunction with Tilson, using techniques of Zeiger [23] and Ginzburg [7]; see also Holcombe [9] for a good exposition with a small correction to Eilenberg's definitions.

When attention is restricted to abstract semigroups (as opposed to transformation semigroups), better methods are available. The *depth decomposition* method of Eilenberg and Tilson [20] is known to yield decompositions for abstract semigroups which are at least as short as, and sometimes shorter than, holonomy decompositions. We briefly recall the depth decomposition method; for full details, see Tilson [20].

Recall that a  $\mathcal{J}$ -class is called *essential* if it contains a non-trivial subgroup. The *depth* of an essential  $\mathcal{J}$ -class is the length of the longest chain of essential  $\mathcal{J}$ -classes strictly above it. The *depth* of the semigroup is defined to be the length of the longest chain of essential  $\mathcal{J}$ -classes in the semigroup, that is, one more than the greatest depth of an essential  $\mathcal{J}$ -class, Let n denote the depth of the semigroup S. For each essential  $\mathcal{J}$ -class J, let  $G_J$ 

denote the maximal subgroup of J. Now for every integer  $0 \le i < n$ , let  $K_i$  be the direct product over all essential  $\mathcal{J}$ -classes of depth i of  $G_i$ .

**Theorem 5.1.** (Depth Decomposition Theorem, Eilenberg-Tilson 1976) Let S be a finite semigroup of depth n, and let  $K_0, \ldots, K_{n-1}$  be as defined above. Then there exist aperiodic monoids  $A_0, \ldots A_n$  such that S divides the wreath product

$$A_n \wr K_{n-1} \wr A_{n-1} \wr \cdots \wr K_0 \wr A_0$$
.

Thus, the depth decomposition theorem gives, for any finite semigroup S, a Krohn-Rhodes decomposition with group length equal to the depth of S. To apply the depth decomposition theorem, we need some information about the  $\mathcal{J}$ -class structure and maximal subgroups of our semigroups. The following proposition provides a description; various parts of it have been observed before [1, 16, 17, 22] but for completeness we prove the entire statement.

**Proposition 5.2.** Let n be a positive integer and k a finite field. Then

- (i)  $T_n(k)$  has depth n-1 if  $k = \mathbb{Z}_2$ , or depth n otherwise. For  $0 \le i \le n-2$  or  $0 \le i \le n-1$  as appropriate,  $T_n(k)$  has  $\binom{n}{i}$  essential  $\mathcal{J}$ -classes of depth i, each of which has maximal subgroup isomorphic to  $T_{n-i}^*(k)$ ;
- (ii)  $UT_n(k)$  has depth n-1. For  $0 \le i < n-1$ ,  $UT_n(k)$  has  $\binom{n}{i}$  essential  $\mathcal{J}$ -classes of depth i, each of which has maximal subgroup isomorphic to  $UT_{n-1}^*(k)$ ; and
- (iii)  $PT_n(k)$  has depth n-1. For  $0 \le i < n-1$ ,  $PT_n(k)$  has  $\binom{n}{i}$  essential  $\mathcal{J}$ -classes of depth i, each of which has maximal subgroup isomorphic to  $PT_{n-i}^*(k)$ .

*Proof.* We begin with the case of  $T_n(k)$ . By Proposition 3.2, the regular  $\mathcal{J}$ -classes are exactly the  $\mathcal{J}$ -classes of the subidentites. Moreover, if e and f are two subidentities, it is easily seen (for example, by using Proposition 3.1), that e is  $\mathcal{J}$ -below f if and only if ef = fe = e. Thus, the lattice of regular  $\mathcal{J}$ -classes is isomorphic to the lattice  $\{0,1\}^n$ , that is to the subset lattice of the set  $\{1,\ldots,n\}$ . In particular, there are  $\binom{n}{i}$  regular  $\mathcal{J}$ -classes at depth i for  $i \in \{0,\ldots,n\}$ .

Now let  $e \in T_n(k)$  be a subidentity at depth i, so that e has rank n-i. It is easily seen that  $eT_n(k)e$  is isomorphic to  $T_{n-i}(k)$  via the map that removes from a matrix all rows and columns for which e has a zero in the corresponding diagonal position. Thus the maximal subgroup at e is isomorphic to  $T_{n-i}^*(k)$ .

Hence, in the case that  $k \neq \mathbb{Z}_2$ , all regular  $\mathcal{J}$ -classes except for that of 0 are essential, giving the required result. In the case that  $k = \mathbb{Z}_2$ , however,  $T_1^*(k)$  is trivial and so there are no essential  $\mathcal{J}$ -classes of depth n-1. Thus, in this case, the depth of the semigroup is one less.

The case of the unitriangular semigroup  $UT_n(k)$  is exactly the same except that the maximal subgroup of the  $\mathcal{J}$ -class of a subidentity with n-i diagonal entries is isomorphic to the unitriangular group  $UT_{n-i}^*(k)$ . However, since  $UT_1^*(k)$  is trivial regardless of the field k, there are never essential  $\mathcal{J}$ -classes of depth n-1, so the semigroup has depth n-1.

For the projective triangular semigroups  $PT_n(k)$ , Proposition 3.3 tells us that the lattice of  $\mathcal{J}$ -classes is the same as that of  $T_n(k)$ ; the maximal subgroup of the  $\mathcal{J}$ -classes of a subidentity of rank n-i is clearly the projective image  $PT_{n-1}^*(k)$  of  $T_{n-i}^*(k)$ . In particular,  $PT_1^*(k)$  is trivial so as in the unitriangular case there are no essential  $\mathcal{J}$ -classes of depth n-1, and the semigroup has depth n-1.

Proposition 5.2 supplies the information needed to apply the Depth Decomposition Theorem to our semigroups. Doing so, we obtain:

$$T_{n}(k) \prec A_{n} \wr T_{1}^{*}(k)^{n} \wr A_{n-1} \wr T_{2}^{*}(k)^{\binom{n}{2}} \wr \cdots \wr T_{n}^{*}(k) \wr A_{0}$$

$$UT_{n}(k) \prec B_{n-1} \wr UT_{2}^{*}(k)^{\binom{n}{2}} \wr B_{n-2} \wr UT_{3}^{*}(k)^{\binom{n}{3}} \wr \cdots \wr UT_{n}^{*}(k) \wr B_{0}$$

$$PT_{n}(k) \prec C_{n-1} \wr PT_{2}^{*}(k)^{\binom{n}{2}} \wr C_{n-2} \wr PT_{3}^{*}(k)^{\binom{n}{3}} \wr \cdots \wr PT_{n}^{*}(k) \wr C_{0}$$

for some aperiodic semigroups  $A_0, \ldots, A_n, B_0, \ldots, B_{n-1}, C_0, \ldots, C_{n-1}$ . Thus, depth decomposition gives alternative (by Theorem 3.4, optimal) decompositions of group length n-1 for  $UT_n(k)$  and  $PT_n(k)$  and a (suboptimal) group length n decomposition for  $T_n(k)$ . The theorem as stated does not give an explicit description of the aperiodic terms; however, the interested reader could compute appropriate ones through an analysis of the proof [20].

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